

The Geometry of Self-dual 2-forms

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Abstract

We show that self-dual 2-forms in $2n$ dimensional spaces determine a $n^2 - n + 1$ dimensional manifold \mathcal{S}_{2n} and the dimension of the maximal linear subspaces of \mathcal{S}_{2n} is equal to the (Radon-Hurwitz) number of linearly independent vector fields on the sphere S^{2n-1} . We provide a direct proof that for n odd \mathcal{S}_{2n} has only one-dimensional linear submanifolds. We exhibit $2^c - 1$ dimensional subspaces in dimensions which are multiples of 2^c , for $c = 1, 2, 3$. In particular, we demonstrate that the seven dimensional linear subspaces of \mathcal{S}_8 also include among many other interesting classes of self-dual 2-forms, the self-dual 2-forms of Corrigan, Devchand, Fairlie and Nuyts and a representation of \mathcal{Cl}_7 given by octonionic multiplication. We discuss the relation of the linear subspaces with the representations of Clifford algebras.

1. Introduction

The self-dual Yang-Mills fields in four dimensions have remarkable properties that have found several physical applications. On the other hand the notion of self-duality cannot be easily generalised to higher dimensions. Here we present a characterisation of (anti)self-dual Yang-Mills fields by an eigenvalue criterion. The main idea is given in our previous paper ^[1]. Here we study the geometry of the space of self dual 2-forms.

Let M be a $2n$ dimensional differentiable manifold, and E be a vector bundle over M with standard fiber R^n and structure group G . A Yang-Mills potential can be represented by a \mathcal{G} valued connection 1-form A on E , where \mathcal{G} is a linear representation of the Lie algebra of the gauge group G . Then the gauge fields are represented by the curvature F of the connection A that is given locally by the \mathcal{G} valued 2-form

$$F = dA - A \wedge A.$$

The Yang-Mills action is the L_2 norm of the curvature 2-form F

$$\|F\|^2 = \int_M \text{tr}(F \wedge *F)$$

where $*$ denotes the Hodge dual defined relative to a positive definite metric on M . The Yang-Mills equations

$$d_E F = 0, \quad *d_E^* F = 0,$$

where d_E is the bundle covariant derivative and $-*d_E^*$ is its formal adjoint, determine the critical points of the action. In $d = 4$ dimensions F is called self-dual or anti-self-dual provided

$$*F = \pm F.$$

Self-dual or anti-self-dual 2-forms are the global extrema of the Yang-Mills action. This can be seen as follows: The Yang-Mills action has a topological lower bound

$$\|F\|^2 \geq \int_M \text{tr}(F \wedge F).$$

The term $\text{tr}(F \wedge F)$ is related to the Chern classes of the bundle. Actually if E is a complex 2-plane bundle with $c_1(E) = 0$, then the topological bound is proportional to $c_2(E)$ and this lower bound is realised by a (anti)self-dual connection. Furthermore, $SU(2)$ bundles over a four manifold are classified

by $\int c_2(E)$, hence self-dual connections are minimal representatives of the connections in each equivalence class of $SU(2)$ bundles. This is a generalisation of the fact that an $SU(2)$ bundle admits a flat connection if and only if it is trivial.

In the literature essentially three notions of self-duality in higher dimensions were being used.

i) A 2-form F in dimension $2n$ is called self-dual if the Hodge dual of F is proportional to F^{n-1} . (Here wedge product of F 's should be understood.) This notion is introduced by Trautman ^[2], and Thirakian ^[3] and used widely by others. For details we refer to a review by Ivanova and Popov ^[4].

ii) Self-dual 2-forms F in dimensions $2n = 4k$ are defined to be the ones such that F^k is self-dual in the Hodge sense. That is $*F^k = \pm F^k$. This is a nonlinear set of conditions and the action which is minimised is

$$\int_M \text{tr}(F^k \wedge * F^k).$$

This notion is adopted by Grossman, Kephart and Stasheff (GKS) in their study of instantons in eight dimensions ^[5].

iii) Both the criteria above are non-linear. Alternatively, (anti)self-dual 2-forms in $2n$ dimensions can be defined as eigen-bivectors of a completely antisymmetric fourth rank tensor that is invariant under a subgroup of $SO(2n)$. The set of such self-dual 2-forms would span a linear space. This notion of self-duality is introduced by Corrigan, Devchand, Fairlie and Nuyts (CDFN) who studied the first-order equations satisfied by Yang-Mills fields in spaces of dimension greater than four and derived $SO(7)$ self-duality equations in \mathbf{R}^8 ^[6].

It can be shown that the self-dual 2-forms defined by the above criteria satisfy Yang-Mills equations. However, the corresponding Yang-Mills action need not be extremal. In order to derive topological bounds in higher dimensions we note that the local curvature 2-form depends on a trivialization of the bundle, but its invariant polynomials σ_k defined by

$$\det(I + tF) = \sum_{k=0}^n \sigma_k t^k$$

are invariant of the local trivialization. We recall that σ_k 's are closed $2k$ -forms, hence they define the deRham cohomology classes in H^{2k} . Furthermore these cohomology classes depend only on the bundle. They are the Chern classes of the bundle E up to some multiplicative constants. It is

also known that the $2k$ -form σ_k can be obtained as linear combinations of $tr F^k$ where F^k means the product of the matrix F with itself k times, with the wedge multiplication of the entries.

In four dimensions the topological bound we wrote above is the only one that is available. In eight dimensions on the other hand it is possible to introduce two independent topological bounds. The topological lower bound on the action

$$\int_M tr(F^2 \wedge^* F^2) \geq k \int_M p_2(E)$$

corresponds to the choice of Thcrakian and GKS. The self-duality (in the Hodge sense) of F^2 gives global minima of this action involving the second Pontryagin number $\int p_2(E)$. In our previous paper ^[1] we introduced another topological lower bound on the action

$$\int tr(F \wedge^* F)^2 \geq k' \int_M p_1(E)^2.$$

This involves the square of the first Pontryagin number and has to be taken into account as the topology of the Yang-Mills bundle on an eight manifold has to be characterised by both the first and the second Pontryagin numbers.

The notion of self-duality introduced by us ^[1] encompasses all the criteria given above. We recall here that a self-dual 2-form can be defined by an eigenvalue criterion in the following way. (We adopt a different terminology, and use self-dual rather than strongly self-dual as it is used in Ref.[1]) Suppose F is a real 2-form in $2n$ dimensions, and let Ω be the corresponding $2n \times 2n$ skew-symmetric matrix with respect to some local orthonormal basis. Then by a change of basis, Ω can be brought to the block-diagonal form

$$\begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 0 & \lambda_n \\ & & & & & -\lambda_n & 0 \end{pmatrix}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Ω . The 2-form F is called self-dual or anti-self-dual provided the absolute values of the eigenvalues are all equal , that is

$$|\lambda_1| = |\lambda_2| = \dots = |\lambda_n|.$$

To distinguish between the two cases, orientation must be taken into account. We define F to be self-dual, if Ω can be brought with respect to an orientation-preserving basis change to the above block-diagonal form such that $\lambda_1 = \lambda_2 = \dots = \lambda_n$. Similarly, we define F to be anti-self-dual, if Ω can be brought to the same form by an orientation-reversing basis change. It is not difficult to check that in dimension $D=4$, the above definition coincides with the usual definition of self-duality in the Hodge sense.

We have already shown that the definition of self-duality by the equality of the eigenvalues implies the criteria (i) and (ii), and the CDFN 2-forms in eight dimensions are self-dual in the above sense.

Let \mathcal{S}_{2n} be the set of self-dual 2-forms in $2n$ dimensions. In Section 2 we give the manifold structure of \mathcal{S}_{2n} . In Section 3, we show that the dimension of maximal linear spaces of \mathcal{S}_{2n} is equal to the number of linearly independent vector fields on S^{2n-1} . We give a direct proof that in eight dimensions starting from the self-duality condition on eigenvalues we obtain the CDFN self-dual 2-form. We also explain the construction of new families of self-dual 2-forms in \mathcal{S}_8 in terms of Clifford representations using octonionic multiplication.

2. The Geometry of Self-dual 2-forms.

In this section we describe the geometrical structure of self-dual 2-forms in arbitrary even dimensions. I denotes an identity matrix of appropriate dimension.

Definition 1. Let \mathbf{A}_{2n} be the set of antisymmetric matrices in $2n$ dimensions. Then $\mathcal{S}_{2n} = \{A \in \mathbf{A}_{2n} \mid A^2 + \lambda^2 I = 0, \lambda \in \mathbf{R}, \lambda \neq 0\}$.

Note that if $A \in \mathcal{S}_{2n}$, and $A^2 = 0$, then $A = 0$, and if $A \in \mathcal{S}_{2n}$, then $\lambda A \in \mathcal{S}_{2n}$ for $\lambda \neq 0$.

Proposition 2. *The set \mathcal{S}_{2n} is diffeomorphic to $(O(2n) \cap \mathbf{A}_{2n}) \times \mathbf{R}^+$.*

Proof. Let $A \in \mathcal{S}_{2n}$ with $A^2 + \lambda^2 I = 0$. Note that $\lambda^2 = -\frac{1}{2n} \text{tr} A^2$. Define $\kappa = [-\frac{1}{2n} \text{tr} A^2]^{1/2}$, $\tilde{A} = \frac{1}{\kappa} A$. Then, $\tilde{A}^2 + I = 0$, hence $\tilde{A} \tilde{A}^\dagger = I$. Consider the map $\varphi : \mathcal{S}_{2n} \rightarrow (O(2n) \cap \mathbf{A}_{2n}) \times \mathbf{R}^+$ defined by $\varphi(A) = (\tilde{A}, \kappa)$. The map φ is 1-1, onto and differentiable. Its inverse is given by $(B, \alpha) \rightarrow \alpha B$ is also differentiable, hence φ is a diffeomorphism. e.o.p.

Remark 3. $O(2n) \cap \mathbf{A}_{2n}$ is a fibre bundle over the sphere S^{2n-2} with fibre $O(2n-2) \cap \mathbf{A}_{2n-2}$. (See Steenrod, Ref.[7])

For our purposes the following description of \mathcal{S}_{2n} is more useful.

Proposition 4. \mathcal{S}_{2n} is diffeomorphic to the homogeneous manifold $(O(2n) \times \mathbf{R}^+)/U(n) \times \{1\}$, and $\dim \mathcal{S}_{2n} = n^2 - n + 1$.

Proof. Let G be the product group $O(2n) \times \mathbf{R}^+$, where \mathbf{R}^+ is considered as a multiplicative group. G acts on \mathcal{S}_{2n} by $(P, \alpha) \cdot A = \alpha(P^t A P)$, where $P \in O(2n)$, $\alpha \in \mathbf{R}^+$, $A \in \mathcal{S}_{2n}$, and t indicates the transpose. Since all matrices in \mathcal{S}_{2n} are conjugate to each other up to a multiplicative constant, this action is transitive, and actually any $A \in \mathcal{S}_{2n}$ can be written as $A = \lambda P^t J P$, where $\lambda = [-\frac{1}{2n} \text{tr} A^2]^{1/2}$, with $P \in O(2n)$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. It can be seen that the isotropy subgroup of G at J is $U(n)$ [8] and $G/U(n)$ is diffeomorphic to \mathcal{S}_{2n} (Ref.[9] p.132, Thm.3.62) Then $\dim \mathcal{S}_{2n} = \dim(O(2n) \times \mathbf{R}^+/U(n))$ can be easily computed as $\dim \mathcal{S} = \dim O(2n) + 1 - \dim U(n) = (2n^2 - n + 1) - n^2 = n^2 - n + 1$. e.o.p.

In particular, in eight dimensions, \mathcal{S}_8 is a 13 dimensional manifold.

As $O(2n)$ has two connected components ($SO(2n)$ and $O(2n) \setminus SO(2n)$), $U(n)$ is connected and $U(n) \subset SO(2n)$, \mathcal{S}_{2n} has two connected components. One of them (that contains J) consists of the self-dual forms and the other of the anti-self-dual forms.

3. Maximal linear submanifolds of \mathcal{S}_{2n}

In this section we will show that the dimension of maximal linear subspaces of \mathcal{S}_{2n} is equal to the number of linearly independent vector fields on S^{2n-1} . The maximal number of pointwise linearly independent vector fields on the sphere S^N is given by the Radon-Hurwitz number k . If $N + 1 = 2n = (2a + 1)2^{4d+c}$ with $c = 0, 1, 2, 3$, then $k = 8d + 2^c - 1$ (See e.g. Ref. [10], p.45, Thm.7.2). This construction gives three vector fields on S^3 , seven on S^7 , three on S^{11} , eight on S^{15} and so on. In particular there is only one vector field on the spheres S^{2n-1} for odd n .

Let \mathcal{L}_{2n}^α be a maximal linear subspace of \mathcal{S}_{2n} , where α is a real parameter. Since the elements of \mathcal{L}_{2n}^α are skew-symmetric and non-degenerate, the dimension of \mathcal{L}_{2n}^α is less than or equal to $2n - 1$. For example in dimension eight \mathcal{S}_8 is 13 dimensional, and we will show that the maximal linear subspaces are 7 dimensional, hence they form six dimensional families.

Proposition 5. *The dimension of the maximal linear subspaces of \mathcal{S}_{2n} is equal to the number of linearly independent vector fields on S^{2n-1} .*

Proof. We will show that the bases of linear subspaces of \mathcal{S}_{2n} give rise to

linearly independent vector fields on S^{2n-1} . Let $\{h_i\}$, $i = 1, \dots, k$ be an orthogonal basis for \mathcal{L}_{2n}^α . That is the h_i 's are linearly independent matrices satisfying $\text{tr}(h_i^t h_j) = \delta_{ij}$. Suppose ξ_1, \dots, ξ_{2n} are coordinates on R^{2n} and let $R = (\xi_1, \dots, \xi_{2n})$ be the radial vector in R^{2n} . Define the vector fields $X_i = h_i R$. Then X_i 's are the tangent vector fields to S^{2n-1} , since $\langle X_i, R \rangle = R^t h_i R = 0$, by the skew-symmetry of h_i 's. The linear independence of h_i 's implies the linear independence of the X_i 's and thus proves our claim: $\sum_{i=1}^k \lambda_i X_i = \sum_{i=1}^k \lambda_i h_i R = 0$ implies $\lambda_1 = \dots = \lambda_k = 0$ because h_i 's are linearly independent.

This shows that the dimension of a maximal linear subspace of \mathcal{S}_{2n} is less than the Radon-Hurwitz number. Conversely, the Radon-Hurwitz number k (associated to $2n$) is equal to the maximal dimension of the Clifford algebra acting on R^{2n} [10]. If we take such a representation of Cl_k on R^{2n} , the images of a generator set $\{v_1, \dots, v_k\}$ (with $v_i^2 = -1$, $v_i v_j + v_j v_i = 0$ for $i \neq j$) are given by skew-symmetric matrices with respect to an appropriate basis of R^{2n} . These images generate linearly a k -dimensional subspace of \mathcal{S}_{2n} . This shows that the dimension of a maximal linear subspace of \mathcal{S}_{2n} is equal to the Radon-Hurwitz number. e.o.p.

This property shows that there is an intimate relationship between generalised self-duality and Clifford algebras. We will give a systematic exposition of this relationship in a subsequent publication.

We remark that X_i 's form an orthogonal frame. As h_i 's and $(h_i + h_j)$'s both belong to \mathcal{S}_{2n} , $h_i^2 = -k_i^2 I$, and $h_i h_j + h_j h_i = k_{ij} I$ for some constants k_i and k_{ij} . Then since $\langle h_i, h_j \rangle = \text{tr}(h_i^t h_j) = 0$ and trace is symmetric, it follows that $h_i h_j + h_j h_i = 0$. Then

$$\begin{aligned} 2\langle X_i, X_j \rangle &= \langle X_i, X_j \rangle + \langle X_j, X_i \rangle \\ &= R^t (h_i^t h_j + h_j^t h_i) R \\ &= -R^t (h_i h_j + h_j h_i) R \\ &= 0. \end{aligned}$$

We now directly prove that for odd n there are no linear subspaces other than the one dimensional ones.

Proposition 6. *Let $\mathcal{M} = \{A \in \mathcal{S} \mid (A + J_o) \in \mathcal{S}\}$. Then $\mathcal{M} = \{kJ \mid k \in \mathbf{R}\}$ for odd n .*

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^t & A_{22} \end{pmatrix}$, where $A_{11} + A_{11}^t = 0$, $A_{22} + A_{22}^t = 0$. As before if $(A + J_o) \in \mathcal{S}$ then $AJ_o + J_o A$ is proportional to the identity.

This gives $A_{11} + A_{22} = 0$ and the symmetric part of A_{12} is proportional to identity. Therefore $A = kJ_o + \begin{pmatrix} A_{11} & A_{12o} \\ A_{12o} & -A_{11} \end{pmatrix}$, where A_{12o} denotes the antisymmetric part of A_{12} and k is a constant. Then the requirement that $A \in \mathcal{S}$ gives

$$[A_{11}, A_{12o}] = 0, \quad A_{11}^2 + A_{12o}^2 + kI = 0, \quad k \in R.$$

As A_{11} and A_{12o} commute, they can be simultaneously diagonalisable, hence for odd n they can be brought to the form

$$A_{11} = \text{diag}(\lambda_1\epsilon, \dots, \lambda_{(n-1)/2}\epsilon, 0)$$

$$A_{12o} = \text{diag}(\mu_1\epsilon, \dots, \mu_{(n-1)/2}\epsilon, 0)$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and 0 denotes a 1×1 block, up to the permutation of blocks. If the blocks occur as shown, clearly $A_{11}^2 + A_{12o}^2$ cannot be proportional to identity. It can also be seen that except for $\lambda_i = \mu_i = 0$ the same result holds for any permutation of the blocks. e.o.p.

4. An explicit construction of linear submanifolds of \mathcal{S}_8

The defining equations of the set \mathcal{S}_8 are homogeneous quadratic polynomial equations for the components of the curvature 2-form and they correspond to differential equations which are quadratic in the first derivative for the connection. Thus the study of their solutions, hence the study of the moduli space of self-dual connections is rather difficult. On the other hand the self-dual 2-forms lying in a linear subspace of \mathcal{S}_{2n} will correspond to linear gauge field equations. The study of the structure of the linear submanifolds of \mathcal{S}_{2n} in general is not attempted here, but at least for \mathcal{S}_8 we know that these linear submanifolds form a 6-parameter family, and there is no a priori reason to single out one of them.

In Ref.[1] we have shown that the 2-forms satisfying a set of 21 equations proposed by Corrigan et al belong to \mathcal{S}_8 . We shall first give a natural way of arriving at them, but it will depend on a reference form. Changing the reference form one obtains translates of this 7-dimensional plane, which in some cases look more pregnant than the original one. Then we shall give a general procedure to construct self-dual 2-forms in $4n$ dimensions using self-dual/anti self-dual forms and certain symmetric matrices in $2n$ dimensions. The matrices corresponding to these building blocks are actually the representations of Clifford algebras in the skew-symmetric matrices

and dual Clifford algebras in symmetric matrices in half dimensions. The CDFN plane, and the representation of \mathcal{Cl}_7 using octonionic multiplication will arise naturally from these constructions.

Note that we excluded the zero matrix from \mathcal{S}_{2n} in our definition in order to obtain its manifold structure. We denote $\overline{\mathcal{S}}_{2n} = \mathcal{S}_{2n} \cup \{0\}$. By linearity of the action of $O(2n)$ on \mathcal{S}_{2n} we obtain the following

Lemma 7. *Let \mathcal{L} be a linear submanifold of $\overline{\mathcal{S}}_{2n}$. Then $\mathcal{L}_P = P^t \mathcal{L} P$, $P \in O(2n)$ is also a linear submanifold of $\overline{\mathcal{S}}_{2n}$.*

Let $J_o = \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon)$, where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that any $A \in \mathcal{S}_8$ is conjugate to J_o , hence any linear subset of $\overline{\mathcal{S}}_8$ can be realized as the translate of a linear submanifold containing J_o under conjugation. Thus without loss of generality we can concentrate on linear subsets containing J_o . In the following, by abuse of notation we will not distinguish between \mathcal{S}_{2n} and its closure.

Proposition 8. *If $A \in \mathcal{S}_8$ and $(A + J_o) \in \mathcal{S}_8$, where $J_o = \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon)$, with $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then*

$$A = \begin{pmatrix} k\epsilon & r_1 S(\alpha) & r_2 S(\beta) & r_3 S(\gamma) \\ -r_1 S(\alpha) & k\epsilon & r_3 S(\gamma') & -r_2 S(\beta') \\ -r_2 S(\beta) & -r_3 S(\gamma') & k\epsilon & r_1 S(\alpha') \\ -r_3 S(\gamma) & r_2 S(\beta') & -r_1 S(\alpha') & k\epsilon \end{pmatrix}$$

where $k \in R$, r_1, r_2, r_3 are in R^+ , and $S(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$, and $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ satisfy

$$\alpha + \alpha' = \beta + \beta' = \gamma + \gamma'$$

.

Proof. If A and $A + J_o$ are both in \mathcal{S}_8 , then the matrix $AJ_o + J_oA$ is proportional to identity. This gives a set of linear equations whose solutions

can be obtained without difficulty to yield

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ -a_{12} & 0 & a_{14} & -a_{13} & a_{16} & -a_{15} & a_{18} & -a_{17} \\ -a_{13} & -a_{14} & 0 & a_{12} & a_{35} & a_{36} & a_{37} & a_{38} \\ -a_{14} & a_{13} & -a_{12} & 0 & a_{36} & -a_{35} & a_{38} & -a_{37} \\ -a_{15} & -a_{16} & -a_{35} & -a_{36} & 0 & a_{12} & a_{57} & a_{58} \\ -a_{16} & a_{15} & -a_{36} & a_{35} & -a_{12} & 0 & a_{58} & -a_{57} \\ -a_{17} & -a_{18} & -a_{37} & -a_{38} & -a_{57} & -a_{58} & 0 & a_{12} \\ -a_{18} & a_{17} & -a_{38} & a_{37} & -a_{58} & a_{57} & -a_{12} & 0 \end{pmatrix}$$

Then the requirement that the diagonal entries in A^2 be equal to each other give the following equations after some algebraic manipulations:

$$a_{13}^2 + a_{14}^2 = a_{57}^2 + a_{58}^2$$

$$a_{15}^2 + a_{16}^2 = a_{37}^2 + a_{38}^2$$

$$a_{17}^2 + a_{18}^2 = a_{35}^2 + a_{36}^2$$

Thus we can parametrise A by

$$a_{13} = r_1 \cos \alpha, \quad a_{15} = r_2 \cos \beta \quad a_{17} = r_3 \cos \gamma$$

$$a_{14} = r_1 \sin \alpha, \quad a_{16} = r_2 \sin \beta \quad a_{18} = r_3 \sin \gamma$$

$$a_{57} = r_1 \cos \alpha', \quad a_{37} = r_2 \cos \beta' \quad a_{35} = r_3 \cos \gamma'$$

$$a_{58} = r_1 \sin \alpha', \quad a_{38} = r_2 \sin \beta' \quad a_{36} = r_3 \sin \gamma'$$

Finally the requirement that the off diagonal terms in A^2 be equal to zero gives quadratic equations, which can be rearranged and using trigonometric identities they give $\alpha + \alpha' = \beta + \beta' = \gamma + \gamma'$. e.o.p.

Thus the set of matrices $A \in \mathcal{S}_8$ such that $(A + J_o) \in \mathcal{S}_8$ constitutes an eight parameter family and the equations of CDFN correspond to the case $\alpha' + \alpha = \beta' + \beta = \gamma' + \gamma = 0$. Thus we have an invariant description of these equations, that we repeat here for convenience.

$$F_{12} - F_{34} = 0 \quad F_{12} - F_{56} = 0 \quad F_{12} - F_{78} = 0$$

$$F_{13} + F_{24} = 0 \quad F_{13} - F_{57} = 0 \quad F_{13} + F_{68} = 0$$

$$F_{14} - F_{23} = 0 \quad F_{14} + F_{67} = 0 \quad F_{14} + F_{58} = 0$$

$$F_{15} + F_{26} = 0 \quad F_{15} + F_{37} = 0 \quad F_{15} - F_{48} = 0$$

$$F_{16} - F_{25} = 0 \quad F_{16} - F_{38} = 0 \quad F_{16} - F_{47} = 0$$

$$\begin{aligned}
F_{17} + F_{28} &= 0 & F_{17} - F_{35} &= 0 & F_{17} + F_{46} &= 0 \\
F_{18} - F_{27} &= 0 & F_{18} + F_{36} &= 0 & F_{18} + F_{45} &= 0
\end{aligned}$$

The (skew-symmetric) matrix of such a 2-form is

$$\begin{pmatrix}
0 & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} & F_{17} & F_{18} \\
& 0 & F_{14} & -F_{13} & F_{16} & -F_{15} & F_{18} & -F_{17} \\
& & 0 & F_{12} & F_{17} & -F_{18} & -F_{15} & F_{16} \\
& & & 0 & -F_{18} & -F_{17} & F_{16} & F_{15} \\
& & & & 0 & F_{12} & F_{13} & -F_{14} \\
& & & & & 0 & -F_{14} & -F_{13} \\
& & & & & & 0 & F_{12} \\
& & & & & & & 0
\end{pmatrix}$$

We will refer to the plane consisting of these forms as the CDFN-plane. Let us now consider as the reference form $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ instead of J_o . J can be obtained from J_o by conjugation $J = P^t J_o P$ with

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Then the conjugation of the CDFN-plane by P is given by the following (D=8 self-dual) 2-form

$$F_{12}J + \begin{pmatrix} \Omega' & \Omega'' \\ \Omega'' & -\Omega' \end{pmatrix}$$

where Ω' is a D=4 self-dual 2-form and Ω'' is a D=4 anti-self-dual 2-form. We found it remarkable that a similar construction was given a long time ago by Witten^[11].

At the end of this section, we shall obtain this plane from a general rule for the construction of orthonormal bases for linear subspaces and also show that it corresponds to the representation of Cl_7 using octonionic multiplication.

We shall now discuss a general procedure for constructing linear subspaces of self-dual forms. Note that \mathcal{S}_{2n} is the set skew-symmetric matrices in $O(2n) \times R$. We define \mathcal{P}_{2n} to be the set of symmetric matrices in $O(2n) \times R$. Recall that an orthonormal basis for a k -dimensional linear subspaces of \mathcal{S}_{2n} corresponds to the representation of Cl_k in the skew-symmetric matrices. Similarly an orthonormal basis for a k -dimensional linear subspace of \mathcal{P}_{2n} corresponds to a representation of the dual Clifford algebra Cl'_k in the symmetric matrices. These bases will be the building blocks for self-dual forms in the double dimension.

We have already shown that in dimensions $2n = 2(2a + 1)$ the maximal linear subspaces of \mathcal{S}_{2n} were one dimensional. Similarly, in dimensions $2n = 4(2a + 1)$, the dimension of maximal linear subspaces of \mathcal{S}_{2n} are three dimensional. It can be seen that the matrices

$$J_0 = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & I \\ 0 & -I & 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},$$

where I is the identity matrix, form an orthonormal basis for three dimensional linear subspaces of $\mathcal{S}_{4(2a+1)}$.

From now on we consider the self-dual 2-forms in $8n$ dimensions. The matrix of a self-dual form can be written in the form

$$F = \begin{pmatrix} A_a & B_a + B_s \\ B_a - B_s & D_a \end{pmatrix},$$

where the matrices A_a, B_a, D_a 's are anti-symmetrical and B_s is symmetrical. The requirement that F^2 be proportional to the identity matrix gives the following equations:

$$A_a^2 = D_a^2, \quad A_a^2 + B_a^2 - B_s^2 = kI, \quad [B_a, B_s] = 0,$$

$$A_a B_a + B_a D_a = 0, \quad B_a A_a + D_a B_a = 0,$$

$$A_a B_s + B_s D_a = 0, \quad B_s A_a + D_a B_s = 0.$$

Now if we furthermore require that F be build up from the linear subspaces of \mathcal{S}_{4n} and \mathcal{P}_{4n} , then we see that A_a, D_a, B_a, B_s have to be nondegenerate.

We shall give now an explicit construction of various linear subspaces of \mathcal{S}_8 . Let \mathcal{A}^- and \mathcal{A}^+ be orthonormal bases for linear subspaces of \mathcal{S}_{2n} and \mathcal{P}_{2n} , respectively.

In two dimensions we have the following structure.

$$\mathcal{A}^- = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \mathcal{A}_{(1)}^+ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \mathcal{A}_{(2)}^+ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

From the commutation relations it can be seen that the orthonormal bases for linear subspaces of self-dual 2-forms in four dimensions are determined by the choice of B_s . The choice $B_s \in \mathcal{A}_{(1)}^+$ leads to the usual anti self-dual 2-forms, while the choice $B_s \in \mathcal{A}_{(2)}^+$ leads to the self-dual 2-forms. Hence in four dimensions we obtain two different sets of orthonormal bases for linear subspaces of \mathcal{S}_4 . By similar considerations, we obtain seven different bases for linear subspaces of \mathcal{P}_4 . The elements of these bases are listed below:

$$\begin{aligned} a_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & a_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & a_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ b_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & b_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & b_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ c_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & c_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & p_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ p_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & d_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & d_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ q_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & q_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & e_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Using the commutation relations it can be shown that in four dimensions we have the following orthonormal bases for the linear subspaces of \mathcal{S}_4 .

$$\mathcal{A}_{(1)}^- = \{a_1, a_2, a_3\}, \quad \mathcal{A}_{(2)}^- = \{b_1, b_2, b_3\},$$

$$\mathcal{A}_{(1)}^+ = \{I\},$$

$$\begin{aligned} \mathcal{A}_{(2)}^+ &= \{c_1, c_2, e_1\}, & \mathcal{A}_{(3)}^+ &= \{p_1, q_2, d_2\}, & \mathcal{A}_{(4)}^+ &= \{p_2, q_1, d_1\}, \\ \mathcal{A}_{(5)}^+ &= \{c_1, p_1, p_2\}, & \mathcal{A}_{(6)}^+ &= \{c_2, q_2, q_1\}, & \mathcal{A}_{(7)}^+ &= \{e_1, d_2, d_1\}, \end{aligned}$$

Orthonormal bases for linear subspaces of \mathcal{S}_8 can be constructed using the sets given above. For example, the choice $B_s \in \{d_2, p_1, q_2\}$ determines the possible choices for B_a 's, A_a 's and D_a 's and leads to the CDFN plane. On the other hand the choice $B_s = I$ leads to the plane obtained by conjugation given above.

We now show that the basis obtained by choosing $B_s = I$ corresponds to the representation of \mathcal{Cl}_7 using octonionic multiplication. Let us describe an octonion by a pair of quaternions (a, b) . Then the octonionic multiplication rule is $(a, b) \circ (c, d) = (ac - \bar{d}b, da + b\bar{c})$. If we represent an octonion (c, d) by a vector in R^8 , its multiplication by imaginary octonions correspond to linear transformations on R^8 . Using the multiplication rule above, it is easy to see that we have the following correspondences:

$$\begin{aligned} (i, 0) &\rightarrow \begin{pmatrix} b_1 & 0 \\ 0 & -b_1 \end{pmatrix}, & (j, 0) &\rightarrow \begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix}, \\ (k, 0) &\rightarrow \begin{pmatrix} b_3 & 0 \\ 0 & -b_3 \end{pmatrix}, & (0, 1) &\rightarrow \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \\ (0, i) &\rightarrow \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix}, & (0, j) &\rightarrow \begin{pmatrix} 0 & a_2 \\ a_2 & 0 \end{pmatrix}, & (0, k) &\rightarrow \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix}. \end{aligned}$$

Finally, we would like to point out that these constructions can be generalised to dimensions which are multiples of eight, by replacing unit element with identity matrices of appropriate size.

In dimensions which are multiples of 16, one can make use of the property $\mathcal{Cl}_{k+8} = \mathcal{Cl}_k \otimes \mathcal{Cl}_8$ to obtain a \mathcal{Cl}_{k+8} representation on R^{16n} , using an already known representation of \mathcal{Cl}_k on R^n . Hence linear subspaces of \mathcal{S}_{16n} can be obtained from the knowledge of the linear subspaces of \mathcal{S}_n .

5. Conclusion

In this paper we have characterised (anti)self-dual Yang-Mills fields in even dimensional spaces by putting constraints on the eigenvalues of F . The previously known cases of self-dual Yang-Mills fields in four and eight dimensions are consistent with our characterisation. We believe this new approach to self-duality in higher dimensions deserves further study. It

might appear more important to try to understand the totality of the non-linear space of self-dual 2-forms as the choice of a linear subspace of \mathcal{S}_{2n} is a priori incidental. Nevertheless, there are some exceptional linear subspaces, probably the most important being the one in eight dimensions given by octonionic multiplication. In this way the close connection between the self-dual gauge fields in eight dimensions and the octonionic instantons ^{[12],[13],[14]} becomes self-evident.

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